Pdf-04 ① H-3021, Dr. Satish Kumar

Remember: If $f(x)$ is a fn on $[-\pi,\pi]$ then

① Fourier cosine series is
i.e we assume $f(x)$ is an even fn.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{n} a_n \cos nx, \quad a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

② Fourier sine series is
i.e we assume $f(x)$ is an odd function

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

Q1. Find Fourier sine series of

$$f(t) = \begin{cases} t, & 0 < t \le \frac{\pi}{2} \\ \frac{\pi}{2}, & \frac{\pi}{2} < t \le \pi \end{cases} \xrightarrow{\text{replace } t \text{ by } x} f(x) = \begin{cases} x, & 0 < x \le \frac{\pi}{2} \\ \frac{\pi}{2}, & \frac{\pi}{2} < x \le \pi \end{cases}$$

Solution. We know Fourier sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$$

$$b_n = \frac{2}{\pi}\int_{0}^{\frac{\pi}{2}} f(x)\sin nx\,dx + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\frac{\pi}{2}} x\sin nx\,dx + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} \frac{\pi}{2}\sin nx\,dx$$

$$= \frac{2}{\pi}\left[-x\frac{\cos nx}{n}\Big|_{0}^{\frac{\pi}{2}} - \int_{0}^{\frac{\pi}{2}}(-)1.\frac{\cos nx}{n}\,dx\right] + \int_{\frac{\pi}{2}}^{\pi} \sin nx\,dx$$

$$b_n = -\frac{1}{n}\cos\frac{n\pi}{2} + \frac{2}{\pi n}\int_{0}^{\frac{\pi}{2}} \cos nx\,dx - \frac{1}{n}\left[\cos nx\right]_{\frac{\pi}{2}}^{\pi}$$

$$b_n = -\frac{1}{n}\cos\frac{n\pi}{2} + \frac{2}{\pi n^2}\left[\sin nx\right]_0^{\frac{\pi}{2}} - \frac{1}{n}\left[\cos n\pi - \cos\frac{n\pi}{2}\right]$$

$$b_n = \cancel{-\frac{1}{n}\cos\frac{n\pi}{2}} + \frac{2}{\pi n^2}\sin\frac{n\pi}{2} - \frac{1}{n}\cos n\pi + \cancel{\frac{1}{n}\cos\frac{n\pi}{2}}$$

$$b_n = \frac{2}{\pi n^2}\sin\frac{n\pi}{2} - \frac{1}{n}\cos n\pi$$

$\therefore$

$$b_1 = \frac{2}{\pi}\sin\frac{\pi}{2} - \cos\pi \Rightarrow \boxed{b_1 = \frac{2}{\pi}+1}$$

$$b_2 = \frac{2}{4\pi}\cdot 0 - \frac{1}{2}\cdot\cos 2\pi \Rightarrow b_2 = -\frac{1}{2}$$

$$b_3 = \frac{2}{9\pi}\sin\frac{3\pi}{2} - \frac{1}{3}\cos 3\pi$$

$$= -\frac{2}{9\pi} + \frac{1}{3} \qquad \left[\because \sin\frac{3\pi}{2} = -1 \ \& \ \cos 3\pi = -1\right]$$

Required fn is given by

$$\therefore f(x) = \left(\frac{2}{\pi}+1\right)\sin x + \left(-\frac{1}{2}\right)\sin 2x + \left(-\frac{2}{9\pi}+\frac{1}{3}\right)\sin 3x + \cdots$$

or

$$f(t) = \left(\frac{2}{\pi}+1\right)\sin t - \frac{1}{2}\sin 2t + \left(\frac{1}{3}-\frac{2}{9\pi}\right)\sin 3t + \cdots$$

Ans..

② Find Fourier sine series of $f(x) = e^{ax}$, $0 < x < \pi$

Sol: Fourier sine series is $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

$$, \ b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} e^{ax}\sin nx\,dx$$

$$= \frac{2}{\pi}\cdot\frac{1}{a^2+n^2}\left[e^{ax}(a\sin nx - n\cos nx)\right]_0^{\pi}$$

$$b_n = \frac{2}{\pi(a^2+n^2)}\left[e^{a\pi}(0 - n\cos n\pi) - 1(0-n)\right]$$

$$b_n = \frac{2}{\pi(a^2+n^2)}\left[n - n e^{a\pi}(-1)^n\right]$$

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Change of interval and functions with arbitrary period.

In electrical engineering problems or in physical problems, the period of the function is not always 2π but T or 2c. This period must be converted to the length 2π. The independent variable x is also to be changed proportionally.

Let the function $f(x)$ be defined on $[-c, c]$. Now we want to change the function to the period 2π so that we can use the formula for a_0, a_n, b_n.

When 2c is the period then variable is x

if 1 is the period then variable is $\frac{x}{2c}$

$\therefore$ if 2π is the period then variable will be

$$\frac{x}{2c} \cdot 2\pi$$

$$= \frac{\pi x}{c}$$

So take new variable

$$l = \frac{\pi x}{c} \Rightarrow x = \frac{cl}{\pi}$$

$\therefore$ $F(l)$ is a function of period 2π.

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Therefore,

Half range series for the interval $]0,c[$

are

① Fourier cosine series on $]0,c[$ as

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{c}, \quad a_0 = \frac{2}{c}\int_0^c f(x)\,dx$$

$$, \quad a_n = \frac{2}{c}\int_0^c f(x)\cos\frac{n\pi x}{c}\,dx$$

② Fourier sine series on $]0,c[$ as

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{c}$$

where $$b_n = \frac{2}{c}\int_0^c f(x)\sin\frac{n\pi x}{c}\,dx$$

Question. I. Find Fourier cosine series of the following

$$f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 2(2-x), & 1 < x < 2 \end{cases}$$

Solution. Given

$$f(x) = \begin{cases} 2x, & 0 < x < 1. \\ 2(2-x), & 1 < x < 2 \end{cases}$$

We have

$$f(x) = \frac{1}{2}a_0 + \sum a_n \cos\frac{n\pi x}{c} + \sum b_n \sin\frac{n\pi x}{c}$$

$$, \quad a_0 = \frac{2}{\pi}\int_0^c f(x)\,dx, \quad a_n = \frac{2}{\pi}\int_0^c f(x)\cos\frac{n\pi x}{c}\,dx$$

$$, \quad b_n = \frac{2}{\pi}\int_0^c f(x)\sin\frac{n\pi x}{c}\,dx$$

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Here $c = 2$

$$a_0 = \frac{2}{2}\int_0^2 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx .$$

$$a_0 = \int_0^1 2x\,dx + \int_1^2 2(2-x)\,dx$$

$$= 1 + 2\left[2x - \frac{x^2}{2}\right]_1^2 = 1+2\left[(2) - \left(2-\frac{1}{2}\right)\right]$$

$$= 1 + 2\left[2-2+\frac{1}{2}\right] = 2 \Rightarrow \boxed{a_0 = 2}$$

$$a_n = \frac{2}{2}\int_0^2 f(x)\cos\frac{n\pi x}{2}\,dx$$

$$= \int_0^1 f(x)\cos\frac{n\pi x}{2}\,dx + \int_1^2 f(x)\cos\frac{n\pi x}{2}\,dx$$

$$= \int_0^1 2x\cos\frac{n\pi x}{2}\,dx + \int_1^2 2(2-x)\cos\frac{n\pi x}{2}\,dx$$

$$a_n = 2\left[\frac{2x}{n\pi}\cdot\sin\left(\frac{n\pi x}{2}\right)\Big|_0^1 - \int_0^1 \frac{2}{n\pi}\sin\frac{n\pi x}{2}\,dx\right] + 2\left[\frac{2(2-x)}{n\pi}\sin\frac{n\pi x}{2}\Big|_1^2 - \int_1^2 (-1)\frac{2}{n\pi}\sin\frac{n\pi x}{2}\,dx\right]$$

$$a_n = 2\left[\frac{2}{n\pi}\sin\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\left(\cos\frac{n\pi x}{2}\right)_0^1\right] + 2\left[0 - \frac{2}{n\pi}\sin\frac{n\pi}{2}\right] - \frac{8}{n^2\pi^2}\left[\cos\frac{n\pi x}{2}\right]_1^2$$

$$a_n = \frac{4}{n\pi}\sin\frac{n\pi}{2} + \frac{8}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{8}{n^2\pi^2} - \frac{4}{n\pi}\sin\frac{n\pi}{2} - \frac{8}{n^2\pi^2}\left[\cos n\pi - \cos\frac{n\pi}{2}\right]$$

$$a_n = \cancel{\frac{4}{n\pi}\sin\frac{n\pi}{2}} + \frac{8}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{8}{n^2\pi^2} - \cancel{\frac{4}{n\pi}\sin\frac{n\pi}{2}} - \frac{8}{n^2\pi^2}\cos n\pi + \frac{8}{n^2\pi^2}\cos\frac{n\pi}{2}$$

$$a_n = \frac{8\times 2}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{8}{n^2\pi^2} - \frac{8}{n^2\pi^2}\cos n\pi$$

$$a_1 = \frac{16}{\pi^2}\times 0 - \frac{8}{\pi^2} - \frac{8}{n^2\pi^2}(1)$$

Take $b_n = 0$ as we are asked only cosine series

$$\therefore f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{2},\quad a_n \text{ is just calculated above.}$$

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So $F(l) = f\left(\frac{cl}{\pi}\right) = \frac{a_0}{2} + a_1 \cos l + a_2 \cos 2l + \cdots$
$+ b_1 \sin l + b_2 \sin 2l + \cdots$

where

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} F(l)\,dl = \frac{1}{\pi}\int_0^{2\pi} f\left(\frac{cl}{\pi}\right) dl$$

but $l = \frac{\pi x}{c}$

$dl = \frac{\pi}{c}\cdot dx$

& $\frac{cl}{\pi} = x$

$$\therefore a_0 = \frac{1}{\pi}\int_0^{2c} f(x)\cdot\frac{\pi}{c}\,dx$$

$$\boxed{a_0 = \frac{1}{c}\int_0^{2c} f(x)\,dx}$$

Similarly $a_n = \frac{1}{\pi}\int_0^{2\pi} F(l)\cos nl = \frac{1}{\pi}\int_0^{2\pi} f\left(\frac{cl}{\pi}\right)\cos nl$

$$\Rightarrow a_n = \frac{1}{\pi}\int_0^{2c} f(x)\,\frac{\pi}{c}\cos\left(n\cdot\frac{\pi x}{c}\right) dx$$

$$\Rightarrow \boxed{a_n = \frac{1}{c}\int_0^{2c} f(x)\cdot\cos\left(\frac{n\pi x}{c}\right) dx}$$

, $x = \frac{cl}{\pi}$

and $$\boxed{b_n = \frac{1}{c}\int_0^{2c} f(x)\sin\left(\frac{n\pi x}{c}\right) dx.}$$

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Excercise

① obtain Fourier cosine series of

$$f(x) = \sin\left(\frac{\pi x}{l}\right), \quad 0 < x < l$$

Hint. $a_0 = \frac{2}{l}\int_0^l \sin\left(\frac{\pi x}{l}\right)dx$, $a_n = \frac{2}{l}\int_0^l \sin\frac{\pi x}{l}\cos\frac{n\pi x}{l}\,dx$

Ans: $f(x) = \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{1}{3}\cos\frac{2\pi x}{l} + \frac{1}{15}\cos\frac{4\pi x}{l} + \frac{1}{35}\cos\frac{6\pi x}{l} + \cdots\right]$

② Find Fourier series of

$$f(x) = \begin{cases} \frac{1}{2}+x, & -\frac{1}{2} < x \leq 0 \\ \frac{1}{2}-x, & 0 < x < \frac{1}{2} \end{cases}$$

Here $2c = 1 \Rightarrow c = \frac{1}{2}$

Hint. $a_0 = \frac{1}{c}\int_{-c}^{c} f(x)\,dx = 2\int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)\,dx$

$a_n = 2\int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)\cos 2n\pi x$, $b_n = 2\int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)\sin 2n\pi x\,dx$

Ans $f(x) = \frac{1}{4} + \frac{2}{\pi^2}\left[\frac{\cos 2\pi x}{1^2} + \frac{\cos 6\pi x}{3^2} + \frac{\cos 10\pi x}{5^2} + \cdots\right]$

③ Find the Fourier series of

$$f(x) = \begin{cases} 2, & -2 \leq x \leq 0 \\ x, & 0 < x < 2 \end{cases}$$

Hint. Here $2c = 4 \Rightarrow c = \frac{4}{2} = 2$

$a_0 = \frac{1}{c}\int_{-c}^{c} f(x)\,dx = \frac{1}{2}\int_{-2}^{2} f(x)\,dx$

$a_n = \frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi x}{c}\,dx$, $b_n = \frac{1}{c}\int_{-c}^{c} f(x)\sin\frac{n\pi x}{c}\,dx$.

Ans. $f(x) = \frac{3}{2} - \frac{4}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \cdots\right]$
$- \frac{2}{\pi}\left[\frac{1}{1}\sin\frac{\pi x}{2} + \frac{1}{2}\sin\frac{2\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} + \cdots\right]$

(4) Find the Fourier Series of

$$f(x) = \begin{cases} -a & , -c < x < 0 \\ a & , 0 < x < c \end{cases}$$

Ans. $f(x) = \frac{4a}{\pi}\left[\sin\frac{\pi x}{c} + \frac{1}{3}\sin\frac{3\pi x}{c} + \frac{1}{5}\sin\frac{5\pi x}{c} + \cdots\right]$

(5) Find half range Sine Series of

$$f(x) = 2x - 1, \quad 0 < x < 1.$$

Ans. $f(x) = -\frac{2}{\pi}\left[\sin 2\pi x + \frac{1}{2}\sin 4\pi x + \frac{1}{3}\sin 6\pi x + \cdots\right]$

(6) Find Fourier Series of

$$f(x) = \begin{cases} -2 & , -4 < x < -2 \\ x & , -2 < x < 2 \\ 2 & , 2 < x < 4 \end{cases}$$

Here $2c = 8$

$c = 4$

Ans : $f(x) = \frac{4}{\pi} + \frac{8}{\pi^2}\sin\frac{\pi x}{4} - \frac{2}{\pi}\sin\frac{2\pi x}{4} + \cdots$

(7) Find Fourier of $f(x) = (x^2 - 2)$, $x \in]-2, 2[$.

Ans. $f(x) = -\frac{2}{3} - \frac{16}{\pi^2}\left[\cos\frac{\pi x}{2} - \frac{1}{4}\cos\pi x + \frac{1}{9}\cos\frac{3\pi x}{2} + \cdots\right]$

(8) If $f(x) = e^{-x}$, $x \in]-c, c[$

show

$$f(x) = (e^{c} - e^{-c})\left\{\frac{1}{2c} - c\left(\frac{1}{c^2+\pi^2}\cos\frac{\pi x}{c} - \frac{1}{c^2+4\pi^2}\cos\frac{2\pi x}{c} + \cdots\right)\right.$$
$$\left. - \pi\left[\frac{1}{c^2+\pi^2}\sin\frac{\pi x}{c} - \frac{2}{c^2+4\pi^2}\sin\frac{2\pi x}{c} + \cdots\right]\right.$$

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State Parseval formula and prove it.

Statement. Let $f(x)$ be a periodic function on $[-c, c]$ then prove

$$\int_{-c}^{c} f^2(x)\,dx = c\left[\frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2+b_n^2)\right]$$

Proof. We know fourier formula as

$$f(x) = \frac{1}{2}a_0 + \sum a_n \cos\frac{n\pi x}{c} + \sum b_n \sin\frac{n\pi x}{c} \quad \text{——— (1)}$$

multipling (1) by $f(x)$ and integrating from $x=-c$ to $x=c$, we have

$$\int_{-c}^{c} f^2(x)\,dx = \frac{1}{2}a_0\int_{-c}^{c} f(x)\,dx + \sum a_n \int_{-c}^{c} f(x)\cos\frac{n\pi x}{c}\,dx + \sum b_n \int_{-c}^{c} f(x)\sin\frac{n\pi x}{c}\,dx \quad \text{—— (2)}$$

We know that

$$a_0 = \frac{1}{c}\int_{-c}^{c} f(x)\,dx \Rightarrow \int_{-c}^{c} f(x)\,dx = c a_0 \quad \text{— (3)}$$

$$a_n = \frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi x}{c}\,dx \Rightarrow \int_{-c}^{c} f(x)\cos\frac{n\pi x}{c}\,dx = c a_n \quad \text{— (4)}$$

$$b_n = \frac{1}{c}\int_{-c}^{c} f(x)\sin\frac{n\pi x}{c}\,dx \Rightarrow \int_{-c}^{c} f(x)\sin\frac{n\pi x}{c}\,dx = c b_n \quad \text{— (5)}$$

So from equation (2), (3), (4) & (5), we get

$$\int_{-c}^{c} f^2(x)\,dx = \frac{1}{2}a_0[c a_0] + \sum a_n(c a_n) + \sum b_n(c b_n)$$

$$= c\left[\frac{1}{2}a_0^2\right] + c\left[\sum a_n^2 + \sum b_n^2\right]$$

, proved.

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Note. (1) If $0 < x < 2c$ or $-c < x < c$

then $$\int_0^{2c} f^2(x)\,dx = c\left[\frac{a_0^2}{2} + \sum a_n^2 + \sum b_n^2\right]$$

(2) If $0 < x < c$ (Half range cosine series)

$$\int_0^c f^2(x)\,dx = \frac{c}{2}\left[\frac{a_0^2}{2} + \sum a_n^2\right]$$

(3) If $0 < x < c$ (Half range sine series)

$$\int_0^c f^2(x)\,dx = \frac{c}{2}\left[\sum b_n^2\right]$$

(4) $$R.M.S = \sqrt{\left(\frac{\int_a^b f^2(x)\,dx}{b-a}\right)}$$

Examples

(1) Using sine series for $f(x) = 1$ in $0 < x < \pi$

show $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$

Soln. We know the sine series is

$$f(x) = \sum b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx = \frac{2}{\pi}\int_0^{\pi} 1.\sin nx\,dx$$

$$= -\frac{2}{\pi n}\left[\cos nx\right]_0^{\pi} = -\frac{2}{\pi n}\left[\cos n\pi - 1\right]$$

$$b_n = -\frac{2}{\pi n}\left[(-1)^n - 1\right]$$

$b_n = 0$ if n is even & $b_n = \frac{4}{\pi n}$ if n is odd.

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$\therefore$ $f(x) = \frac{a_0}{2} + \sum a_n \cos nx + \sum b_n \sin nx$

where $a_n = 0$, $a_0 = 0$

$\Rightarrow 1 = \sum b_n \sin nx$

$1 = b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + b_4 \sin 4x + \cdots$

$1 = \frac{4}{\pi} \sin x + \frac{4}{3\pi} \sin 3x + \frac{4}{5\pi} \sin 5x + \cdots$

We know Perseval's formula for sine series is

$$\int_0^{\pi} f^2(x)\,dx = \frac{\pi}{2}\left[\sum b_n^2\right]$$

$$\therefore \int_0^{\pi} 1 \cdot dx = \frac{\pi}{2}\left[b_1^2 + b_3^2 + b_5^2 + \cdots\right]$$

$$\pi = \frac{\pi}{2}\left[\left(\frac{4}{\pi}\right)^2 + \left(\frac{4}{3\pi}\right)^2 + \left(\frac{4}{5\pi}\right)^2 + \cdots\right]$$

$$\Rightarrow 2 = \frac{16}{\pi^2}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right]$$

$$\Rightarrow \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots.$$

Q2. If $f(x) = \begin{cases} \pi x, & 0 < x < 1 \\ \pi(2-x), & 1 < x < 2 \end{cases}$

Using half range cosine series, show

$$\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots = \frac{\pi^4}{96}.$$

Solⁿ. We know half range cosine series is

$$f(x) = \frac{1}{2}a_0 + \sum a_n \cos\frac{n\pi x}{c}, \quad c = 2, \text{ Here}$$

where $a_0 = \frac{2}{c}\int_0^c f(x)\,dx = \frac{2}{2}\left[\int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx\right]$

$$= \int_0^1 \pi x\,dx + \int_1^2 \pi(2-x)\,dx$$

$$a_0 = \frac{\pi}{2} + \pi\left(2x - \frac{x^2}{2}\right)_1^2 = \frac{\pi}{2} + \pi\left(2 - \frac{3}{2}\right) = \pi$$

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$$a_n = \frac{2}{c}\int_0^c f(x)\cos\frac{n\pi x}{c}\,dx = \frac{2}{2}\int_0^2 f(x)\cos\frac{n\pi x}{2}\,dx$$

$$a_n = \int_0^1 f(x)\cos\frac{n\pi x}{2}\,dx + \int_1^2 f(x)\cos\frac{n\pi x}{2}\,dx$$

$$a_n = \int_0^1 \pi x\cos\frac{n\pi x}{2}\,dx + \int_1^2 \pi(2-x)\cos\frac{n\pi x}{2}\,dx$$

$$a_n = \pi\left[x\cdot\frac{2}{n\pi}\sin\frac{n\pi x}{2}\Big|_0^1 - \int_0^1 1\cdot\frac{2}{n\pi}\sin\frac{n\pi x}{2}\,dx\right]$$

$$+\pi\left[(2-x)\frac{2}{n\pi}\sin\frac{n\pi x}{2}\Big|_1^2 - \int_0^1(-1)\frac{2}{n\pi}\sin\frac{n\pi x}{2}\,dx\right]$$

$$= \pi\left[\frac{2}{n\pi}\sin\frac{n\pi}{2} + \frac{2}{n\pi}\left(\frac{2}{n\pi}\right)\cos\frac{n\pi x}{2}\Big|_0^1\right]$$

$$+\pi\left[0 - \frac{2}{n\pi}\sin\frac{n\pi}{2} + \frac{2}{n\pi}(-)\frac{2}{n\pi}\cos\frac{n\pi x}{2}\Big|_1^2\right]$$

$$= \pi\left[\frac{2}{n\pi}\sin\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\left(\cos\frac{n\pi}{2}-1\right)\right]$$

$$+\pi\left[-\frac{2}{n\pi}\sin\frac{n\pi}{2} - \frac{4}{n^2\pi^2}\left(\cos n\pi - \cos\frac{n\pi}{2}\right)\right]$$

$$= \pi\left[\frac{4}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{4}{n^2\pi^2} - \frac{4}{n^2\pi^2}\cos n\pi + \frac{4}{n^2\pi^2}\cos\frac{n\pi}{2}\right]$$

$$= \pi\left[\frac{8}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{4}{n^2\pi^2} - \frac{4}{n^2\pi^2}\cos n\pi\right]$$

$$a_n = \frac{4}{n^2\pi}\left[2\cos\frac{n\pi}{2} - 1 - \cos n\pi\right]$$

putting $n = 1, 2, 3, 4, \ldots$, we get

$a_1 = 0$, $a_2 = -\frac{4}{\pi}$, $a_3 = 0$, $a_4 = 0$, $a_5 = 0$, $a_6 = \frac{-4}{9\pi}$.

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So by Perseval's formula, we have

$$\int_0^c f^2(x)\,dx = \frac{c}{2}\left[\frac{a_0^2}{2} + a_1^2 + a_2^2 + a_3^2 + \cdots\right]$$

$$\int_0^1 f^2(x)\,dx + \int_1^2 f^2(x)\,dx = \frac{2}{2}\left[\frac{\pi^2}{2} + 0 + \left(\frac{-4}{\pi}\right)^2 + 0^2 + 0^2 + 0^2 + \left(\frac{-4}{9\pi}\right)^2 + \cdots\right]$$

$$\Rightarrow \int_0^1 (\pi x)^2\,dx + \int_1^2 \pi^2(2-x)^2\,dx = \frac{\pi^2}{2} + \frac{16}{\pi^2} + \frac{16}{81\pi^2} + \cdots$$

$$\Rightarrow \frac{\pi^2}{3} + \pi^2(-)\left.\frac{(2-x)^3}{3}\right|_1^2 = \frac{\pi^2}{2} + \frac{16}{\pi^2} + \frac{16}{81\pi^2} + \cdots$$

$$\Rightarrow \frac{\pi^2}{3} - \pi^2\left(0 - \frac{1}{3}\right) = \frac{\pi^2}{2} + \frac{16}{\pi^2} + \frac{16}{81\pi^2} + \cdots$$

$$\Rightarrow \frac{2\pi^2}{3} - \frac{\pi^2}{2} = \frac{16}{\pi^2}\left[1 + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right]$$

$$\Rightarrow \frac{\pi^2}{6} = \frac{16}{\pi^2}\left[1 + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right]$$

$$\Rightarrow \frac{\pi^4}{96} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\cdots$$ Ans.

(3). Prove that for $0 < x < \pi$

(a) $x(\pi - x) = \frac{\pi^2}{6} - \left[\frac{\cos 2x}{1^2} + \frac{\cos 4x}{2^2} + \frac{\cos 6x}{3^2} + \cdots\right]$

(b) $x(\pi - x) = \frac{8}{\pi}\left[\frac{\sin x}{1^2} + \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} + \cdots\right]$

Deduce from (a) & (b) respectively

(c) $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$ (d) $\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi}{945}$

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Soln. Given $f(x) = x(\pi - x)$, $0 < x < \pi$

Half range cosine series is

$$f(x) = \frac{1}{2}a_0 + \sum a_n \cos nx$$

Where

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\,dx$$

$$= \frac{2}{\pi}\left[\pi\frac{x^2}{2} - \frac{x^3}{3}\right]_0^{\pi} = \frac{2}{\pi}\left[\frac{\pi^3}{2} - \frac{\pi^3}{3}\right]$$

$$a_0 = \frac{2}{\pi}\left[\frac{\pi^3}{6}\right] = \frac{\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi}(\pi x - x^2)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[(\pi x - x^2)\frac{\sin nx}{n}\Big|_0^{\pi} - \int_0^{\pi}(\pi - 2x)\frac{\sin nx}{n}\,dx\right]$$

$$= \frac{2}{\pi}\left[-\frac{1}{n}\left\{(\pi - 2x)\cdot(-)\frac{\cos nx}{n}\right\}\Big|_0^{\pi} + \frac{1}{n}\int_0^{\pi}(-2)\frac{\cos nx}{n}\,dx\right]$$

$$= \frac{2}{\pi}\left[-\frac{1}{n}\left\{\frac{\pi\cos n\pi}{n} + \frac{\pi}{n}\right\} - \frac{2}{n^3}(\sin nx)_0^{\pi}\right]$$

$$a_n = \frac{2}{n}\left[-\frac{\cos n\pi}{n} - 1\right] = -\frac{2}{n}\left[\frac{(-1)^n + 1}{n}\right]$$

$a_n = -\frac{4}{n^2}$ if n is even, $a_n = 0$ if n is odd.

$$\therefore f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx \Rightarrow$$

$$\pi x - x^2 = \frac{1}{2}\left(\frac{\pi^2}{3}\right) + a_2\cos 2x + a_4\cos 4x + a_6\cos 6x + \cdots$$

$$\pi x - x^2 = \frac{\pi^2}{6} - 4\left[\frac{\cos 2x}{2^2} + \frac{\cos 4x}{4^2} + \cdots\right] \quad [\because a_n = 0 \text{ if } n \text{ is odd}]$$

Perseval's formula $\Rightarrow \frac{2}{\pi}\int_0^{\pi} f^2(x)\,dx = \frac{a_0^2}{2} + \sum a_n^2$

$$\frac{2}{\pi}\int_0^{\pi}(\pi x - x^2)^2\,dx = \frac{1}{2}\left(\frac{\pi^2}{6}\right)^2 + \sum_{n=1}^{\infty}\frac{16}{n^4}$$

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$$\frac{2}{\pi}\int_0^{\pi}(\pi^2x^2+x^4-2\pi x^3)\,dx = \frac{1}{2}\left(\frac{\pi^4}{9}\right) + 16\left(\frac{1}{2^4}+\frac{1}{4^4}+\frac{1}{6^4}+\cdots\right)$$

$$\Rightarrow \frac{2}{\pi}\left[\frac{\pi^2}{3}\cdot\pi^3+\frac{1}{5}\cdot\pi^5-\frac{2}{4}\pi\,\pi^4\right] = \frac{\pi^4}{18}+16\left[\frac{1}{2^4}+\frac{1}{4^4}+\frac{1}{6^4}+\cdots\right]$$

$$\Rightarrow \frac{2}{\pi}\left[\frac{\pi^5}{3}+\frac{\pi^5}{5}-\frac{1}{2}\pi^5\right] = \frac{\pi^4}{18}+\left[\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots\right]$$

$$\Rightarrow \frac{2\pi^5}{\pi}\left[\frac{1}{3}+\frac{1}{5}-\frac{1}{2}\right] = \frac{\pi^4}{18}+\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}$$

$$\Rightarrow 2\pi^4\left(\frac{10+6-15}{30}\right) = \frac{\pi^4}{18}+\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}$$

$$\Rightarrow \frac{\pi^4}{15} = \frac{\pi^4}{18}+\sum_{n=1}^{\infty}\frac{1}{n^4} \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{15}-\frac{\pi^4}{18} = \frac{\pi^4}{90}.$$

i.e $\boxed{\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}}$ (a) part is proved.

(b) Half range sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}(\pi x - x^2)\sin nx\,dx$$

$$b_n = \frac{2}{\pi}\left[(\pi x - x^2)(-)\frac{\cos nx}{n}\Big|_0^{\pi} + \int_0^{\pi}(\pi-2x)\frac{\cos nx}{n}\,dx\right]$$

$$= \frac{2}{\pi}\left[0+\frac{(\pi-2x)}{n}\cdot\frac{\sin nx}{n}\Big|_0^{\pi} - \int_0^{\pi}\frac{1}{n}\cdot(-2)\frac{\sin nx}{n}\,dx\right]$$

$$= \frac{2}{\pi}\left[\frac{2}{n^2}\int_0^{\pi}\sin nx\,dx\right] = \frac{4}{\pi n^2}(-)\left(\frac{\cos nx}{n}\right)_0^{\pi}$$

$$b_n = -\frac{4}{\pi n^3}\left[(-1)^n-1\right] \Rightarrow b_n = 0 \text{ if } n \text{ is even}$$

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$b_n = \frac{8}{n^3\pi}$ if n is odd

$\therefore \pi x - x^2 = b_1 \sin x + b_3 \sin 3x + b_5 \sin 5x + \cdots$

$\left[\because b_n = 0 \text{ if } n \text{ is even}\right]$

$$\pi x - x^2 = \frac{8}{\pi}\left[\frac{\sin x}{1^3} + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \cdots\right]$$

By Parseval's formula

$$\frac{2}{\pi}\int_0^{\pi} f^2(x)\,dx = \sum_{n=1}^{\infty} b_n^2 \quad , n \text{ is odd}$$

$$\frac{2}{\pi}\int_0^{\pi} (\pi x - x^2)^2\,dx = \sum_{n=1}^{\infty} b_n^2 \quad , n \text{ is odd}$$

$$\frac{\pi^2}{15} = \sum\left\{\frac{8}{\pi}\left[\frac{1}{n^3}\right]\right\}^2 \quad , n \text{ is odd}$$

$$\Rightarrow \frac{\pi^2}{15} = \sum_{n=1}^{\infty} \frac{64}{\pi^2}\cdot\frac{1}{n^6} \Rightarrow \frac{\pi^4}{15\times 64} = \sum_{n=1}^{\infty} \frac{1}{n^6} \text{ , n is odd}$$

$$\Rightarrow \frac{\pi^4}{960} = \frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \cdots$$

Let $S = \frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \cdots$

$$S = \left(\frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \cdots\right) + \left(\frac{1}{2^6} + \frac{1}{4^6} + \frac{1}{6^6} + \cdots\right)$$

$$S = \frac{\pi^4}{960} + \frac{1}{2^6}\left(\frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \cdots\right)$$

$$S = \frac{\pi^4}{960} + \frac{S}{64} \Rightarrow \left(S - \frac{S}{64}\right) = \frac{\pi^4}{960}$$

$$\Rightarrow \frac{63}{64} S = \frac{\pi^4}{960}$$

$$\Rightarrow S = \frac{\pi^4 \times \cancel{64}\,4}{63 \times \cancel{960}\,60} \Rightarrow S = \frac{\pi^4}{945}$$